UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3401**

ASSESSMENT PATTERN : **MATH3401A**

MODULE NAME : **Methods Of Mathematical Physics I**

DATE : **02-May-08**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2007/08-MATH3401A-001-EXAM-22

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Use the method of variation of parameters to obtain a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}\ln x.$$

(b) Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the complex t-plane so that

$$\int_{C_n} e^{xt} f(t)dt \qquad (n = 1, 2)$$

are non-trivial solutions of the differential equation

$$x\frac{d^2y}{dx^2} - (4x-1)\frac{dy}{dx} + (4x-2)y = 0.$$

Hence find one of the solutions in closed form.

2. The equation of motion of a body moving in a straight line is

$$\ddot{x} + f(\dot{x}) + g(x) = 0$$

with $f(x)$ and $g(x)$ being regular functions of x and the dot denoting differentiation with respect to the variable t. Obtain the differential equation in the phase plane and show that

(a) periodic motion of the body occurs if and only if the corresponding phase trajectory is closed,

(b) periodic motion is impossible if $\dot{x}f(\dot{x})$ has constant sign.

Consider the equation of motion

$$\ddot{x} + \dot{x} - x^2 + x = 0.$$

Determine the nature of the singular points and sketch the trajectories in the phase plane.

3. Consider the equation

$$\ddot{x} + x - \epsilon\left(\dot{x}^3 - \dot{x}\right) = 0$$

where a dot denotes differentiation with respect to the variable t. Show that this can be written as

$$n^2\frac{d^2x}{d\theta^2} + x - \epsilon n\left(\frac{dx}{d\theta}\right)^3 + \epsilon n\frac{dx}{d\theta} = 0$$

where $x(\theta)$ is 2π-periodic with frequency $\frac{n}{2\pi}$.

By seeking solutions of the form

$$x(\theta) = x_0(\theta) + \epsilon x_1(\theta) + \epsilon^2 x(\theta) + \dots\ ,$$

$$n = n_0 + \epsilon n_1 + \epsilon^2 n_2 + \dots\ ,$$

with $0 < \epsilon \ll 1$, show that if $x(0) = 0$ then a *periodic* solution is given by

$$x = \frac{2}{\sqrt{3}}\sin(\theta) + \epsilon\left[\frac{1}{4}\left(\frac{1}{3}\right)^{\frac{3}{2}}\left[\cos(\theta) - \cos(3\theta)\right] + B_1\sin(\theta)\right]$$

and

$$n = 1 + \epsilon^2 n_2$$

where B_1 and n_2 are unknown constants.

Derive the differential equation which holds at $O(\epsilon^2)$.

HINT: $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

4. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0,$$

where $\epsilon > 0$ is a constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\sin[t + \phi(t)]$ if

$$\dot{A} = -\epsilon f\left(A\sin\chi, A\cos\chi\right)\cos\chi,$$

$$\dot{\phi} = \epsilon A^{-1} f\left(A\sin\chi, A\cos\chi\right)\sin\chi,$$

where $\chi = \phi + t$.

If $0 < \epsilon \ll 1$, describe a method for finding approximate solutions of these equations.

For the case of $f(x, \dot{x}) = (4x^2 - 1)\,\dot{x}$ and $A(0) = A_0 > 1$, $\phi(0) = \phi_0$, show that the amplitude of the periodic solution is 1.

Show also that the general solution $x(t)$ is given approximately by

$$x = \left(1 - \left[1 - \frac{1}{A_0^2}\right]e^{-\epsilon t}\right)^{-\frac{1}{2}}\sin\left[t + \phi_0\right].$$

Deduce the limit-cycle solution for $x(t)$.

[You may assume that $\int_0^{2\pi} \cos^2\theta d\theta = \pi$ and $\int_0^{2\pi} \cos^4\theta d\theta = \frac{3\pi}{4}$.]

5. State *without proof* a form of Watson's Lemma. Use the lemma to find two leading terms in the asymptotic expansions of each of the following integrals as $x \to +\infty$:

   (a)
$$\int_0^1 t^x \ln(1 - \ln(t))dt,$$

   (b)
$$\int_{\frac{\pi}{2}}^{\pi} e^{-xt} \sin(t) \sinh\left(t - \frac{\pi}{2}\right)dt.$$

   Throughout the interval $a \leq t \leq b$ the function $f(t)$ is continuous, the function $\phi(t)$ is increasing from $t = a$ to $t = b$ and $\phi'(b) \neq 0$. Show that, as $x \to +\infty$,

$$\int_a^b e^{x\phi(t)} f(t)dt \sim \frac{e^{x\phi(b)} f(b)}{\phi'(b)x}.$$

   (You may assume
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \quad ,$$

   and
$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots \quad).$$